The Open University

Faculty of Science, Technology, Engineering and Mathematics
M208 Pure mathematics

M208

TMA 06 2019J

(Covers Book F) **Cut-off date 9 April 2020**

You can submit your TMA either by post to your tutor or electronically as a PDF file by using the University's online TMA/EMA service.

Before starting work on the TMA, please read the document *Student guidance for preparing and submitting TMAs*, available from the 'Assessment' tab of the M208 website.

In the wording of the questions:

- *write down*, *list* or *state* means 'write down without justification' (unless otherwise stated)
- *find*, *determine*, *calculate*, *derive*, *evaluate* or *solve* means 'show all your working'
- *prove*, *show*, *deduce* or *verify* means 'justify each step'
- *sketch* means 'sketch without justification' and *describe* means 'describe without justification' (both unless otherwise stated).

In particular, if you use a definition, result or theorem to go from one line to the next, then you must state clearly which fact you are using – for example, you could quote the relevant unit and page, or give a Handbook reference. Remember that when you use a theorem, you must demonstrate that all the conditions of the theorem are satisfied.

The number of marks assigned to each part of a question is given in the right-hand margin, to give you a rough indication of the amount of time that you should spend on each part.

Your work should be written in a good mathematical style, as demonstrated by the exercise and worked exercise solutions in the study texts. You should explain your solutions carefully, using appropriate notation and terminology, defining any symbols that you introduce, and writing in proper sentences. Five marks (referred to as good mathematical communication, or GMC, marks) on this TMA are allocated for how well you do this.

Your score out of 5 for GMC will be recorded against Question 6. (You do not have to submit any work for this particular question.)

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3.1

You should read the information on the front page of this booklet before you start working on the questions.

Question 1 (Unit F1) – 23 marks

(a) In this part of the question you may assume any results that are proved in Unit F1, but you should refer to any result that you use.

Prove each of the following statements.

(i) $\lim_{x \to 1} \frac{2x^2 - x - 1}{x^3 + 2x^2 - x - 2} = \frac{1}{2}$ [4]

(ii) $\lim_{x \to 0} \frac{e^{x^3} - 1}{x^3} = 1$ [4]

(iii) $\lim_{x \to \infty} \frac{3e^x - 2\log x}{x^4 + e^x} = 3$ [4]

(b) Prove that the function

$$f(x) = \sin\left(\frac{4}{x^2}\right)$$

does not tend to a limit as x tends to 0. [5]

(c) Use the ε-δ definition of continuity to prove that the function

$$f(x) = 2x^2 - 3x + 1$$

is continuous at the point $c = 2$. [6]

Question 2 (Unit F2) – 17 marks

(a) Prove from the definition of differentiability that the function

$$f(x) = \frac{x+1}{x^2+2}$$

is differentiable at the point 2, and determine $f'(2)$. [4]

(b) Sketch the graph of the function

$$f(x) = \begin{cases} 1 + x, & x < 0, \\ \cos x, & x \geq 0. \end{cases}$$

Use a result or rule from the module to determine whether f is differentiable at the point 0. [4]

(c) Let

$$f(x) = x^5 + 3x - 2 \quad (x \in \mathbb{R}).$$

(You may ASSUME that $f(\mathbb{R}) = \mathbb{R}$.)

(i) Prove that f has an inverse function f^{-1} that is differentiable on $\mathbb{R}$.

(ii) Verify that $f^{-1}(2) = 1$, and determine the value of $(f^{-1})'(2)$. [5]

(d) Use the Mean Value Theorem to show that if $f(x) = x\cos x$, then there is a point $c \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ such that $f'(c) = -\frac{1}{\sqrt{2}}$. [4]

Question 3 (Unit F2) – 10 marks

This question concerns the following exercise.

Exercise

Prove that the limit

$$\lim_{x\to 0} \frac{e^x \sin(4x)}{x^2 + 2\sinh x}$$

exists, and determine its value.

(a) Explain why the following solution is incorrect, identifying at least three errors or omissions. (There may be more than three errors or omissions, but you are required to identify only three. Your three errors or omissions should not include incorrect statements or omissions that follow entirely sensibly from earlier errors.) [5]

Solution (incorrect!)

Let

$$f(x) = e^x \sin(4x) \quad \text{and} \quad g(x) = x^2 + 2\sinh x, \quad x \in \mathbb{R}.$$

Then $f(0) = 0$ and $g(0) = 0$.

Also,

$$f'(x) = e^x \sin(4x) + 4e^x \cos(4x) \quad \text{and} \quad g'(x) = 2x + 2\cosh x,$$
$$x \in \mathbb{R}.$$

Hence the limit $\lim\limits_{x\to 0} \dfrac{e^x \sin(4x)}{x^2 + 2\sinh x}$ exists and equals

$$\lim_{x\to 0} \frac{e^x \sin(4x) + 4e^x \cos(4x)}{2x + 2\cosh x}.$$

Substituting $x = 0$ in the numerator and denominator, we get that this limit, and so the original limit, equals $\dfrac{0+4}{0+2} = 2$.

(b) Give a correct solution to the exercise, justifying your answer carefully. [5]

Question 4 (Unit F3) – 20 marks

(a) Let

$$f(x) = \begin{cases} 2, & x = 0, \\ x, & 0 < x \le 3, \\ 9 - 2x, & 3 < x \le 4. \end{cases}$$

Sketch the graph of f, and evaluate $L(f, P)$ and $U(f, P)$ for each of the following partitions P of $[0, 4]$.

(i) $P = \{[0, 2], [2, 4]\}$

(ii) $P = \{[0, 1], [1, 2], [2, 3], [3, 4]\}$ [6]

(b) Let

$$I_n = \int_0^{\pi/2} \sin(2x) x^n \, dx, \quad \text{for } n \ge 2.$$

(i) By using integration by parts twice, or otherwise, show that a reduction formula for the integral I_n is

$$I_n = \frac{1}{2}\left(\frac{\pi}{2}\right)^n - \frac{1}{4} n(n-1) I_{n-2}, \quad \text{for } n \ge 2.$$ [6]

(ii) Evaluate I_0. [1]

(iii) Deduce the value of I_2. [1]

(c) Prove that

$$\frac{\pi}{3(1+\sqrt{3}\,)} \le \int_{\pi/6}^{\pi/3} \frac{1 + 2\sin x}{1 + 2\cos x} \, dx \le \frac{\pi}{12}(1 + \sqrt{3}\,).$$ [6]

Question 5 (Unit F4) – 25 marks

(a) Calculate the Taylor polynomial $T_2(x)$ at 1 for the function

$$f(x) = \frac{2x}{1 + x}.$$

Show that $T_2(x)$ approximates $f(x)$ with an error of less than 0.002 on the interval $[1, 1.25]$. [9]

(b) (i) Determine the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n(n+1)} (x+2)^n.$$ [8]

(ii) State the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n(n+1)^2} (x+2)^{n+1},$$

and give a brief explanation of your answer. [2]

(c) Use the General Binomial Theorem to determine the first four terms of the Taylor series at 0 for the function

$$f(x) = (1 - 3x)^{4/3}.$$

State the radius of convergence of this power series. [6]

Question 6 (Book F) – 5 marks

Five marks on this assignment are allocated for good mathematical communication in your answers to Questions 1 to 5.

You do not have to submit any extra work for Question 6, but you should check through your assignment carefully, making sure that you have explained your reasoning clearly, used notation correctly and written in proper sentences. [5]